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On the Substructure Synthesis Method

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Substructure synthesis is a model reduction method whereby a complex structure is regarded as an assemblage of substructures. The motion of each of the substructures is represented by a series of admissible functions (vectors), and the substructures are made to act as a single structure by imposing certain geometric compatibility conditions at the boundary between any two adjacent substructures. Because the series of admissible functions (vectors) represents the motion of the substructures only approximately and because the geometric compatibility conditions are only approximations of the true conditions, the computed eigensolution is only an approximation of the actual one. Convergence of computed eigenvalues to actual eigenvalues requires two limiting processes, one in which the number of admissible functions (vectors) is increased and the other in which the number of geometric compatibility conditions is increased.

Introduction

SUBSTRUCTURE synthesis is a modeling method permitting the representation of a relatively complex structure by a reduced number of degrees of freedom. The concept of substructure synthesis can be traced to Hurty.[1,2] The general idea is to regard a given structure as an assemblage of substructures acting together in some fashion. The motion of each substructure is represented by a set of substructure modes, referred to by Hurty as "component modes." Hence, substructure synthesis can be regarded as an application of the Rayleigh-Ritz method to complex structures. Substructure modes are obtained by solving an eigenvalue problem for a given substructure. Computationally, this represents a much more desirable problem than solving the full eigenvalue problem for the assembled structure. Then, the substructure can be represented by a reduced number of lower substructure modes. To guarantee that the various substructures do not act independently but as parts of the whole structure, Hurty introduced the concept of constraint modes. Generally, the constraint modes enforce geometric compatibility at a preselected set of points on the "internal boundary" between any two adjacent substructures. In addition, Hurty uses rigid-body modes, so that component modes are measured relative to a displaced position defined by the rigid-body modes and constraint modes. The rigid-body modes and constraint modes are defined in terms of displacements of a given number of points on the substructure boundaries. In reality, there is no need to distinguish between rigid-body modes and constraint modes, an observation made by Craig and Bampton,[3] who suggested that they all be regarded as constraint modes. One of the questions arising in connection with Hurty's method is how to select the set of boundary points. Craig and Bampton model substructures by the finite-element method, which permits ready identification of the constraint displacements as nodal displacements at the boundaries.

Since publication of the original work by Hurty, there have been many suggestions for improvement. For the most part, they are directed toward two basic questions: 1) how to select a set of substructure modes and 2) how to enforce geometric compatibility at internal boundaries. In Hurty's method, the modes used to represent the substructure motion are essentially "fixed-constraint modes." Benfield and Hruda[4] advocate substructure modes that are not necessarily constrained and can be free-free. They allow for improvement in substructure modes by using stiffness and inertial loadings at internal boundaries, thus attempting to account for the effect of adjacent substructures. The use of unconstrained modes is also advocated by Goldman,[5] Hou,[6] and Dowell.[7] To improve the accuracy of methods using either "fixed-constraint modes" or unconstrained modes, Hintz[8] advocates using sets of interface modes that are capable of producing the exact static response of a substructure to static loading on the internal boundaries. The effect of modes not retained explicitly is considered by MacNeal[9] and by Rubin.[10] The procedure described by MacNeal employs hybrid component modes, i.e., modes obtained with some interface coordinates free and others fixed, and uses statically derived modes to improve accuracy. The procedure suggested by Rubin employs residual flexibility to a second order of approximation.

The methods discussed above have one thing in common, namely, they all regard a structure as an assemblage of substructures and they all represent the motion of each substructure as a linear combination of substructure modes. The main difference lies in the kind of modes used, where the modes satisfy a certain substructure eigenvalue problem. But substructure eigenvalue problems cannot be defined uniquely, which accounts for the variety of ideas as to the type of modes to be used. The substructures may be modeled by either discrete or distributed-parameter mathematical models. In the case of distributed-parameter substructures, substructure modes are likely to be difficult, if not impossible, to produce. Hence, the question arises whether it is not possible to represent the motion of substructures by some simple functions, not necessarily substructure modes, such as low-degree polynomials. Because substructure synthesis is some type of Rayleigh-Ritz method, although perhaps not recognized as such, the answer must be affirmative. Indeed, *for a Rayleigh-Ritz analysis it is necessary that substructures be represented by admissible functions only,*[11] as long as the functions are from a complete set. Of course, substructure modes are certainly admissible functions, but the interest lies in simple functions, easy to work with computationally. This is the general idea expounded in some recent papers by these authors.[12,13]

Next, let us consider discrete substructures obtained by the finite element method, or any other suitable method. A characteristic of discrete substructures is that they are likely to possess a very large number of degrees of freedom, where the number may reach into the tens of thousands. Hence, producing discrete substructure modes may require extensive computational effort, so that the question arises whether it is not possible, by analogy with the case of distributed substructures, to represent the motion of discrete substructures

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by a reduced number of "shape vectors" that are easy to produce and easy to work with computationally. Again, the answer is affirmative. The idea of shape vectors is advanced in two works by these authors,[14,15] in which suitable shape vectors are called "admissible vectors." Admissible vectors are the discrete counterpart of admissible functions for distributed-parameter substructures. They must satisfy the geometric boundary conditions of the discrete substructure, and they must form certain geometric patterns, i.e., they must have acceptable shapes, where the latter is the discrete counterpart of differentiability requirements associated with admissible functions. In view of the preceding, the various substructure syntheses based on representation of either distributed or discrete substructures by a set of substructure modes[1-10] can be regarded as special cases of the general substructure synthesis methods of Ref. 13 or 15, respectively.

Without regard to whether a substructure is distributed or discrete, we must distinguish between structures in which adjacent substructures are connected at a single point, at a number of discrete points and at an infinite number of points. Quite frequently, however, actual boundaries consisting of lines and surfaces, i.e., of an infinite number of points, are represented by a finite number of points. For discrete substructures obtained by the finite element method, the finite number of points might consist of nodal points on the internal boundary. The general idea of representing substructure motion by admissible functions or admissible vectors is valid for any type of internal boundaries, so that the main difference between the cases of single-point boundaries and multipoint or infinitely-many points boundaries lies in the satisfaction of the geometric compatibility conditions. Special care must be exercised when the geometric compatibility conditions at internal boundaries cannot be satisfied exactly, as demonstrated later in this paper.

When adjacent substructures are connected at a single point, such as in chain-like structures, the geometric compatibility can be satisfied automatically by a suitable kinematical procedure.[12] The motion is described conveniently in Ref. 12 in terms of rigid-body motions of local reference frames for the substructures and elastic displacements relative to the moving reference frames. This approach is capable of accommodating rotating substructures, which makes it ideally suited for the treatment of gyroscopic systems, such as helicopters and spacecraft with spinning parts.

When adjacent substructures are two- and three-dimensional, however, internal boundaries consist of lines and surfaces, and hence of an infinity of points. In such cases, the kinematical procedure of Ref. 12 is no longer possible and geometric compatibility must be enforced by other means.[13,15] If the substructure is represented by a finite series of admissible functions or a truncated series of admissible vectors, then enforcing the geometric compatibility conditions is tantamount to imposing constraints on the displacement vector of the disconnected structure. In general, it is impossible to satisfy the geometric compatibility conditions at the entire infinity of points on internal boundaries. This would require the satisfaction of an infinite set of constraint equations, which is inconsistent with a model reduction procedure based on a finite set of admissible functions or a reduced set of admissible vectors. This raises the question whether a substructure synthesis that does not satisfy the internal compatibility conditions exactly is a Rayleigh-Ritz method at all. The answer is a qualified yes, as demonstrated later in this paper.

When the actual geometric compatibility conditions at internal boundaries cannot be satisfied exactly, the disconnected structure is joined together by imposing geometric compatibility conditions that are only approximations of the actual ones. We regard the structure that is no longer disconnected, but whose internal boundary conditions are only approximations of the actual ones, as a fictitious structure lying between the totally disconnected structure and the actual structure and we refer to it as an "intermediate structure." Then, we can state that *the Rayleigh-Ritz method is valid for the intermediate structure.*

The approximate geometric compatibility conditions can be enforced in various ways. It is done here by the weighted residual method, using weighting functions for distributed adjacent substructures and weighting vectors for discrete adjacent substructures, so that the intermediate structure represents a mathematical contrivance defined by the type of weighting functions (vectors) used and their number. In the limit, as the number of weighting functions (vectors) increases, the intermediate structure must approach the actual structure. Hence, we must consider two limiting processes, one in which the number of substructure admissible functions (vectors) is increased, a process converging to the intermediate structure, and another in which the number of weighting functions (vectors) is increased, a process converging to the actual structure. The two limiting processes can be carried out sequentially or simultaneously.

The Substructure Synthesis Method

Let us consider a structure consisting of a given number of substructures s. The substructures can have distributed or discrete stiffness and mass properties. First we shall assume that all substructures act independently of one another and then we shall couple the substructures by constraining them to work together as a single structure. We distinguish between external and internal boundaries for a given substructure, where external boundaries coincide with the boundaries of the whole structure and internal boundaries are interfaces with adjacent substructures. Of course, there can be substructures with only internal boundaries. The object is to produce a reduced-order model of the structure permitting reasonably accurate computation of the structure vibration modes.

Considering a substructure s with distributed mass and stiffness and denoting by $\boldsymbol{u}_s(P,t)$ the displacement vector of any point on the substructure relative to an inertial space, where P is the nominal position of the point in question, we can write the kinetic and potential energy in the symbolic form

$$T_s = \tfrac{1}{2}(\sqrt{\rho_s}\dot{\boldsymbol{u}}_s, \sqrt{\rho_s}\dot{\boldsymbol{u}}_s) \tag{1a}$$

$$V_s = \tfrac{1}{2}[\boldsymbol{u}_s, \boldsymbol{u}_s] \tag{1b}$$

where $(\sqrt{\rho_s}\dot{\boldsymbol{u}}_s, \sqrt{\rho_s}\dot{\boldsymbol{u}}_s)$ represents the inner product of $\sqrt{\rho_s}\dot{\boldsymbol{u}}_s$ with itself, in which ρ_s is the mass density, and $[\boldsymbol{u}_s, \boldsymbol{u}_s]$ represents the inner product[11] of the vector $\boldsymbol{u}_s$ with itself. The energy inner product involves spatial derivatives of $\boldsymbol{u}_s$ through order p and both inner products involve integration over the domain of extension D_s of substructure s. The kinetic energy T_s is assumed to be positive definite and the potential energy V_s can be positive definite or only positive semidefinite.

Our interest lies in the representation of the structure by a reduced-order model. To this end, we assume that the displacement vector $\boldsymbol{u}_s$ can be written in the form

$$\boldsymbol{u}_s(P,t) = \phi_s(P)\zeta_s(t) \tag{2}$$

where ϕ_s is a $3 \times N_s$ matrix of admissible functions and ζ_s is an N_s-vector of generalized coordinates. Note that admissible functions must satisfy only the geometric boundary conditions of the disconnected substructure and must be p-times differentiable. In engineering practice, admissible functions are sometimes referred to as "shape functions." Introducing Eq. (2) into Eqs. (1), we obtain

$$T_s = \tfrac{1}{2}\dot{\zeta}_s^T M_s \dot{\zeta}_s \tag{3a}$$

$$V_s = \tfrac{1}{2}\zeta_s^T K_s \zeta_s \tag{3b}$$

where $M_s = (\sqrt{\rho_s}\phi_s, \sqrt{\rho_s}\phi_s)$ is the $N_s \times N_s$ *substructure reduced mass matrix* and $K_s = [\phi_s, \phi_s]$ is the $N_s \times N_s$ *substructure*

reduced stiffness matrix. Note that M_s is a real symmetric positive definite matrix and K_s is a real symmetric positive definite or positive semidefinite matrix.

Next, let us consider the case in which substructure s is discrete, but has a large number n_s of degrees of freedom. Letting $\boldsymbol{u}_s(t)$ be the displacement vector, the kinetic and potential energy of the substructure can be written in the form

$$T_s = \tfrac{1}{2}\dot{\boldsymbol{u}}_s^T m_s \dot{\boldsymbol{u}}_s \tag{4a}$$

$$V_s = \tfrac{1}{2}\boldsymbol{u}_s^T k_s \boldsymbol{u}_s \tag{4b}$$

where m_s and k_s are the $n_s \times n_s$ *substructure mass matrix and stiffness matrix,* respectively. The matrix m_s is real symmetric and positive definite and the matrix k_s is real and symmetric and can be positive definite or positive semidefinite. The object is to represent the discrete substructure by a reduced-order model. To this end, we use the analogy with the distributed system and let

$$\boldsymbol{u}_s(t) = \phi_s \zeta_s(t) \tag{5}$$

in which ϕ_s is an $n_s \times N_s$ matrix with its columns equal to n_s-dimensional *admissible vectors,*[14,15] where $N_s \ll n_s$. Note that admissible vectors are the discrete counterparts of admissible functions. Hence, they must satisfy the geometric boundary conditions of the substructure and they must have acceptable shapes, where the latter condition is the discrete counterpart of the differentiability condition. They should be independent, and linear combinations of the vectors must be capable of representing the lowest eigenvectors of the substructure with reasonable accuracy. By analogy with distributed systems, the admissible vectors can be identified as shape vectors. The analogy can be completed by insisting that as $n_s \to \infty$ an admissible vector must become an admissible function. Introducing Eq. (5) into Eqs. (4), we obtain the $N_s \times N_s$ *substructure reduced mass* and *stiffness matrices,* respectively. Clearly, they are real and symmetric. Hence, whether the substructure is distributed or discrete, it is represented by an N_s-degree-of-freedom discrete system.

In actuality, the distinction between distributed and discrete substructures is not as sharp as may be inferred from the preceding discussion. More often than not the distinction is one of modeling, and it tends to disappear in the reduced-order model, as the reduced kinetic energy and potential energy have the forms (3), regardless of the original model.

Next, let us turn our attention to the synthesis process. If the structure consists of m substructures ($s = 1,2,..,m$) and if these substructures are acting independently of one another, then the *disjoint kinetic energy* and *potential energy* are

$$T_d = \sum_{s=1}^{m} T_s = \tfrac{1}{2}\dot{\zeta}_d^T M_d \dot{\zeta}_d \tag{6a}$$

$$V_d = \sum_{s=1}^{m} V_s = \tfrac{1}{2}\zeta_d^T K_d \zeta_d \tag{6b}$$

where

$$\zeta_d = [\zeta_1^T \vdots \zeta_2^T \vdots \cdots \vdots \zeta_m^T]^T$$

is the N-dimensional *disjoint configuration vector,*

$$N = \sum_{s=1}^{m} N_s$$

M_d is the $N \times N$ block-diagonal *disjoint mass matrix* and K_d the $N \times N$ block-diagonal *disjoint stiffness matrix.* Considering synchronous motion, as in normal mode vibration, and eliminating the time dependence, we can introduce the *disjoint Rayleigh quotient* defined as

$$R_d = \zeta_d^T K_d \zeta_d / \zeta_d^T M_d \zeta_d \tag{7}$$

In reality the substructures act together as a whole structure. The joining together of the disjoint substructures is part of the synthesis process. At external boundaries, geometric compatibility implies in the case of distributed substructures that the displacements and derivatives of the displacements normal to the boundary through order p-1 must conform to the type of support. At internal boundaries, the geometric compatibility must ensure that structural integrity is preserved. For example, considering two adjacent distributed substructures r and s and letting $p = 2$, we must have

$$\boldsymbol{u}_s = \boldsymbol{u}_r, \quad \partial \boldsymbol{u}_s / \partial \boldsymbol{n}_s = -\partial \boldsymbol{u}_r / \partial \boldsymbol{n}_r \text{ over } S_{rs}, r,s = 1,2,...,m; \quad s \neq r \tag{8a,b}$$

where S_{rs} is the internal boundary and $\boldsymbol{n}_s$ and $\boldsymbol{n}_r$ are outward normals to the internal boundary of substructures s and r, respectively. Similarly, in the case of discrete substructures, geometric compatibility implies that the displacements and rotations at internal boundary points shared by substructures r and s must be the same, or

$$(\boldsymbol{u}_s)_I = (\boldsymbol{u}_r)_I, \quad r,s = 1,2,...,m; \quad s \neq r \tag{9}$$

where $(\ )_I$ denotes the entries in the generalized coordinate vector corresponding to the displacements and rotations at the boundary points. Clearly, the possibility exists in which substructure r is distributed and substructure s is discrete. In this case, the displacement and slope associated with the distributed substructure must be matched with the displacements and rotations of the discrete substructure at the boundary points common to both substructures. For example, in the case of displacement matching we have

$$\boldsymbol{u}_r(P_h) = (\boldsymbol{u}_s)_{DI_h}, \quad P_h \in S_{rs} \tag{10a}$$

where the subscript DI_h indicates the displacement part of the vector $\boldsymbol{u}_s$ corresponding to point P_h of the internal boundary S_{rs}. Similarly in the case of slope and rotation matching we have

$$\frac{\partial \boldsymbol{u}_r(P_k)}{\partial \boldsymbol{n}_r} = (\boldsymbol{u}_s)_{RI_k}, \quad P_k \in S_{rs} \tag{10b}$$

where the subscript RI_k indicates the rotation part of the vector $\boldsymbol{u}_s$ corresponding to point P_k on the internal boundary S_{rs}. If both displacements and rotations are matched at P_h, then $k = h$.

Equations (8), (9), or (10), as the case may be, can be used to join together the otherwise disjoint substructures. Indeed, they can be regarded as constraints to be imposed on the disjoint vector ζ_d. Their satisfaction, most likely only approximate satisfaction, results in constraint equations. Hence, if there are M_C such constraint equations, then the whole structure has only $n = N - M_C$ degrees-of-freedom. Denoting by ζ the n vector of independent generalized coordinates for the whole structure, the relation between the disjoint vector ζ_d and the vector ζ can be written in the form

$$\zeta_d = C\zeta \tag{11}$$

where C is an $N \times n$ constraint matrix. An algorithm for producing the matrix C based on Householder transformations can be found in Ref. 16. Introducing Eq. (11) into Eq. (7), we obtain the Rayleigh quotient for the assembled structure

$$R = \zeta^T K \zeta / \zeta^T M \zeta \tag{12}$$

where $K=C^TK_dC$ and $M=C^TM_dC$ are $n\times n$ real symmetric stiffness and mass matrices for the assembled structure.

The approximate natural modes of vibration of the structure can be obtained by rendering the Rayleigh quotient stationary, which amounts to solving the eigenvalue problem

$$KU=MU\Lambda \tag{13}$$

where U is the modal matrix and Λ the diagonal matrix of the eigenvalues for the assembled structure.

Geometric Compatibility and the Intermediate Structure

We wish to consider the problem of geometric compatibility at internal boundaries. Equations (8) require satisfaction of geometric compatibility at every point of the surface S_{rs}, where S_{rs} represents the internal boundary between distributed substructures r and s, and hence at an infinite number of points. The exact satisfaction of Eqs. (8) can be achieved only in special cases. Clearly, they can be satisfied if the surface S_{rs} is not a surface but a single point. They can also be satisfied if S_{rs} is a smooth simple curve, such as a straight line, and the admissible functions in r and s are conformable. This is possible in the application of the finite-element method to a simple problem, where S_{rs} would be the boundary between two elements, but is very unlikely in the case of substructure synthesis, where for the most part the geometry of the substructures is sufficiently diverse to render conformable admissible functions impractical. Hence, in general Eqs. (8) can be satisfied only approximately. In various substructure synthesis methods, such as Hurty's method,[1,2] Eqs. (8) are satisfied at a finite number of preselected points. A different approach is adopted in Ref. 13. More specifically, Eqs. (8) are satisfied only approximately by a weighted residual method. According to this approach, we consider the weighting functions g_{rs_i} and h_{rs_j} and insist that

$$\int_{S_{rs}} g_{rs_i}(\boldsymbol{u}_s-\boldsymbol{u}_r)\,dS_{rs}=\boldsymbol{0} \qquad r,s=1,2,\ldots,m;\quad s\neq r \tag{14a}$$

$$\int_{S_{rs}} h_{rs_j}\left(\frac{\partial \boldsymbol{u}_s}{\partial \boldsymbol{n}_s}+\frac{\partial \boldsymbol{u}_r}{\partial \boldsymbol{n}_r}\right)dS_{rs}=\boldsymbol{0} \qquad r,s=1,2,\ldots,m;\quad s\neq r \tag{14b}$$

If the weighting functions g_{rs_i} and h_{rs_j} form complete sets and the number of members of each set is infinite, then the geometric compatibility conditions (8) are satisfied exactly. This is impossible, however, because $\boldsymbol{u}_r$ and $\boldsymbol{u}_s$ are defined in terms of finite numbers N_r and N_s of admissible functions, respectively, and hence they contain a finite number of undetermined coefficients. It follows that we must use only finite numbers $M_{rs}^{(g)}$ and $M_{rs}^{(h)}$ of weighting functions g_{rs_i} $(i=1,2,\ldots,M_{rs}^{(g)})$ and h_{rs_j} $(j=1,2,\ldots,M_{rs}^{(h)})$, where $M_{rs}^{(g)}+M_{rs}^{(h)}=M_{rs}$. We discuss the nature of the functions g_{rs_i} and h_{rs_j} later in this paper. At this point, however, it is perhaps of interest to point out that if g_{rs_i} and h_{rs_j} are spatial Dirac delta functions, or

$$g_{rs_i}=\delta(P-P_{rs_i}),\quad h_{rs_j}=\delta(P-P_{rs_j}),\quad s,r=1,2,\ldots,m;\quad s\neq r \tag{15a,b}$$

then Eqs. (8) will be satisfied at the individual points $P=P_{rs_i}$ and $P=P_{rs_j}$, respectively. Hence, the various approaches characterized by the satisfaction of the geometric compatibility at a finite number of points can be regarded as special cases of the approach defined by Eqs. (14).

For discrete substructures, the approach is very similar to that for distributed substructures. Assuming that the vectors $(\boldsymbol{u}_s)_I$ and $(\boldsymbol{u}_r)_I$ have dimensions equal to m_{rs}, we can introduce certain weighting vectors $\boldsymbol{g}_{rs_i}$ of the same dimension m_{rs} and write

$$\boldsymbol{g}_{rs_i}^T(\boldsymbol{u}_s-\boldsymbol{u}_r)_I=0, \qquad r,s=1,2,\ldots,m;\quad s\neq r \tag{16}$$

If there are m_{rs} such weighting vectors, then the geometric compatibility conditions (9) are satisfied exactly. The number m_{rs} is expected to be very large, however, so that exact satisfaction of the geometric compatibility conditions is not feasible. Hence, we shall consider only M_{rs} vectors $\boldsymbol{g}_{rs_i}(i=1,2,\ldots,M_{rs})$, where $M_{rs}\ll m_{rs}$, so that Eqs. (9) are satisfied only approximately. Of course, the dimension of the vectors $\boldsymbol{g}_{rs_i}$ remains m_{rs}. By analogy with weighting functions in the form of spatial Dirac delta functions, we observe that if a given $\boldsymbol{g}_{rs_i}$ is a standard unit vector, with all its components equal to zero except for one component that is equal to one, then the corresponding components in $(\boldsymbol{u}_s)_I$ and $(\boldsymbol{u}_r)_I$ are equal to one another. Hence, the displacement in a given direction or the rotation about a given axis are matched at the corresponding point shared by substructures r and s.

If substructure r is distributed and substructure s is discrete, then we consider certain weighting vectors $\boldsymbol{g}_{rs_i}$ $(i=1,2,\ldots,M_{rs})$ and satisfy the geometric compatibility conditions (10) only approximately by insisting that

$$\boldsymbol{g}_{rs_i}^T[\boldsymbol{U}_r-(\boldsymbol{u}_s)_I]=0, \qquad r,s=1,2,\ldots,m;\quad s\neq r \tag{17}$$

where $\boldsymbol{U}_r$ is an m_{rs}-dimensional vector. The vector $\boldsymbol{U}_r$ is formed from the displacement vector $\boldsymbol{u}_r$ for the distributed substructure r by taking the various entries in $\boldsymbol{U}_r$ as the values of $\boldsymbol{u}_r(P_h)$ and $\partial \boldsymbol{u}_r(P_h)/\partial \boldsymbol{n}_r$ at all points P_h of the internal boundary between substructures r and s.

In the preceding developments, the geometric compatibility conditions (8), (9), or (10) at internal boundaries are satisfied only approximately. To examine the implications of this approximation, it is convenient to envision a fictitious structure with internal boundaries such that the preceding boundary conditions, satisfied only approximately in the case of the actual structure, are satisfied exactly in the case of the fictitious structure. We refer to the fictitious structure as an "intermediate structure," because it lies between the totally disjoint structure and the actual structure. Exact compatibility is ensured for an intermediate structure by satisfaction of either Eqs. (14), Eqs. (16), or Eqs. (17) for a given number M_{rs} of weighting functions g_{rs_i} and h_{rs_j} or weighting vectors $\boldsymbol{g}_{rs_i}$. The particular intermediate structure is defined not only by the number M_{rs} but also by the character of the weighting functions or weighting vectors. Because the geometric compatibility is satisfied exactly for the intermediate structure, *the Rayleigh theorem concerning the stationarity of Rayleigh's quotient*[11] *applies to the intermediate structure and not to the actual structure.* As far as the actual structure is concerned, the intermediate structure is subject to fewer constraints, so that *the eigenvalues of the intermediate structures tend to be lower than those of the actual structure.* Hence, Rayleigh's quotient, Eq. (12), does not provide an upper bound for the lowest eigenvalue of the actual structure, and indeed *substructure synthesis can yield computed eigenvalues that are lower than the actual ones.* Of course, because the Rayleigh-Ritz method tends to yield eigenvalues that are higher than the actual ones, lowering of the computed eigenvalues tends to be a move in the right direction, but the implications must be fully recognized. In refining the model to converge to the actual eigenvalues, two limiting processes must be considered, one in which the number of admissible functions for each substructure is increased, and one for which the number of constraint equations, resulting from the enforcement of the geometric compatibility conditions at internal boundaries, is increased. The two limiting processes can be carried out sequentially or simultaneously. Convergence aspects of the limiting processes are discussed in Ref. 13.

Let us point out that even satisfying Eqs. (9) or (10) exactly may not be sufficient in some cases to satisfy the actual geometric compatibility between adjacent substructures exactly. This is because actual compatibility conditions may be defined for an infinite number of points, whereas discrete substructures do not readily permit satisfaction of compatibility conditions at any points other than a finite number of points on the internal boundary. Hence, all previous methods,[1-10] if they satisfy geometric compatibility at only a finite number of points, in essence are replacing the actual structure with an intermediate structure, a fact not brought out explicitly in Refs. 1-10. Of course, if adjacent substructures are modeled by the finite-element method and the elements along the internal boundaries are conformable, then this problem can be eliminated. This would represent a severe restriction on the choice of substructures modeling, and one not imposed in Refs. 1-10.

Selection of Admissible Functions (Vectors) and Weighting Functions (Vectors)

Perhaps the most delicate question in substructure modeling is the selection of admissible functions. Clearly, admissible functions are not unique and convergence can be achieved with different sets of admissible functions. The admissible functions representing the motion of a given substructure need satisfy only the geometric boundary conditions at external boundaries. The geometric compatibility at internal boundaries is enforced through Eqs. (8). Of course, substructure admissible functions must be from a complete set of functions, be linearly independent, and be p times differentiable. The criterion that substructure admissible functions be from a complete set ensures that one can obtain as accurate a representation of a substructure as desired, provided the number of functions in the set is sufficiently large. However, much of the inaccuracy in substructure synthesis can be attributed to errors at the internal boundaries. These errors are the result of piecing together different sets of functions from each substructure. On the other hand, actual eigenfunctions for the assembled structure are likely to be smoother, i.e., be more times differentiable at the internal boundaries than the functions resulting from piecing together substructures admissible functions. Hence, admissible functions for adjacent substructures should be similar in nature, as this permits more accurate representation of the substructure motion at internal boundaries by means of only a small number of functions. The advantage of substructure admissible functions is that they belong to a class of functions appreciably larger than the class of substructure eigenfunctions, no matter how a substructure eigenvalue problem is defined. Hence, they can be chosen as simple functions that are easy to work with computationally, such as low-degree polynomials.

As mentioned earlier, the concept of admissible functions, together with their many advantages, can be extended to discrete substructures of relatively large order by introducing the concept of admissible vectors. Admissible functions must satisfy geometric boundary conditions imposed on the substructure and be p times differentiable. By analogy, admissible vectors must also satisfy the geometric boundary conditions and they must form certain geometric patterns, i.e., they must have acceptable shapes, where the latter is the discrete counterpart of the differentiability conditions. In a pictorial language, admissible vectors can be referred to as shape vectors. The admissible function-admissible vector analogy can be placed into sharper focus by considering that, as the dimension of the discrete substructure approaches infinity, an admissible vector becomes an admissible function. The analogy also forms the basis for the selection of suitable admissible vectors. For a given discrete substructure s with dimension n_s, such as one resulting from the finite-element discretization of a distributed substructure, we can conceive of a simple distributed substructure, not necessarily the same as the original distributed substructure, but one that is similar dynamically to the discrete substructure. Such a similar distributed substructure admits a set of admissible functions $\phi_i(D)$, where D is the domain of the distributed substructure. Next, consider a set of nodal points associated with the finite-element model of the actual substructure, as well as a set of points in the domain D of the similar distributed substructure, so that there is a one-to-one correspondence between the two sets of points. We can now construct a set of n_s-dimensional admissible vectors $\boldsymbol{\phi}_{s_i}$ by discretizing, or sampling in space, the set of admissible functions $\phi_i(D)$. Indeed, the values and the slopes of the admissible functions $\phi_i(D)$ at the set of points chosen can be regarded as the entries of the admissible vectors $\boldsymbol{\phi}_{s_i}$ corresponding to the displacements and rotations at the nodal points of the finite element model. This process guarantees that the admissible vectors $\boldsymbol{\phi}_{s_i}$ possess acceptable shapes. Clearly, if the functions $\phi_i(D)$ are linearly independent, then the admissible vectors obtained by "discretizing" these functions will also be linearly independent.

The idea of constructing admissible vectors by discretizing admissible functions is sufficiently general to encompass all large-dimensional discrete substructures. Discrete substructures are the result of employing a particular modeling technique, such as the finite-element method or the lumped-parameter method, to represent an actual substructure, which is most likely distributed. Although the preceding discussion depended on the definition of a set of finite-element nodal points in space, any other discrete substructure mathematical model that is in terms of displacements and rotations at actual points in space fits into this basic mold. Hence, even a substructure represented by a large number of discrete masses of finite dimensions (i.e., they possess mass moments of inertia) and springs can be represented by admissible vectors, where the admissible vectors are obtained by discretizing admissible functions.

It is perhaps worth emphasizing that substructure modes, i.e., substructure eigenfunctions (eigenvectors), can be used as admissible functions (vectors), although their use is not really necessary, and at times not even desirable. Often stated reasons for using substructure modes are that they represent suitable admissible functions (vectors) and that they can be computed automatically, without human intervention. However, the computational effort necessary to determine substructure modes can be appreciable. This price may not be worth paying when simpler admissible functions (vectors) are capable of yielding equally good results with much less effort and only a minimal amount of human intervention. At the very least, if one insists on using substructure modes, then there is no reason to be unduly concerned over their accuracy.

There remains the question as to how to choose the weighting functions $g_{rs_j}(i=1,2,\ldots,M_{rs}^{(g)})$ and $h_{rs_j}(j=1,2,\ldots,M_{rs}^{(h)})$ in Eqs. (8) and the weighting vectors $\mathbf{g}_{rs_i}(i=1,2,\ldots,M_{rs})$ in Eqs. (9) and (10). Considering first the weighting functions g_{rs_i} and h_{rs_i}, it is obvious that they are required to be defined over and to be integrable on the internal boundary S_{rs} between distributed substructures r and s. To satisfy the geometric compatibility conditions (8) exactly, the geometric compatibility conditions (14a) and (14b) must be used in conjunction with infinite sets of weighting functions g_{rs_i} and h_{rs_j}, respectively, where the infinite sets must be complete. If the weighting functions are not from complete sets, then the intermediate structures do not necessarily converge to the original structure, although approximations to the original structure will still be obtained. In practical applications, the numbers $M_{rs}^{(g)}$ and $M_{rs}^{(h)}$ of weighting functions g_{rs_i} and h_{rs_j} must be taken as finite. Moreover, the choice of finite sets of weighting functions must depend on certain computational requirements. In the first place, we require that for a particular internal boundary surface S_{rs} any finite number $M_{rs}^{(g)}+M_{rs}^{(h)}=M_{rs}$ of geometric compatibility conditions (14) must be linearly independent. This ensures that the solution of the M_{rs} compatibility conditions is unique

and consists of M_{rs} generalized coordinates. The particular set chosen is not necessarily unique, and its identification may be difficult. In order to meet these requirements, the choice of weighting functions g_{rs_i} and h_{rs_j} and the choice of substructure admissible functions, i.e., the entries of the matrices ϕ_r and ϕ_s, cannot always be made independently. For instance, substructure admissible functions and/or their derivatives normal to the internal boundary may be equal to zero along the internal boundary. Such admissible functions can never contribute to the satisfaction of the compatibility conditions (14a). Therefore, in order to satisfy a total of M_{rs} compatibility conditions (14), as defined by particular choices of weighting functions g_{rs_i} and h_{rs_j}, it is necessary to consider a sufficient number of substructure admissible functions so that the compatibility conditions lead to a set of M_C algebraic equations, permitting a solution for M_C of the components of ζ_d in terms of the remaining $n=N-M_C$ components.

The computational requirements in choosing a set of weighting vectors $g_{rs_i}(i=1,2,\ldots,M_{rs})$ in Eqs. (9) and (10) are similar to those for weighting functions. Weighting vectors and substructure admissible vectors must be chosen so that linearly independent compatibility conditions (16) and (17) result. If the weighting vectors are taken as standard unit vectors, then the displacement in a given direction or the rotation about a given axis are matched at the corresponding point shared by substructures r and s. This suggests the possibility that when the internal boundary consists of a large number of nodal points, geometric compatibility between substructures r and s can be approximated with reasonable accuracy by choosing a set of unit weighting vectors so as to satisfy Eqs. (9) or (10) exactly at a preselected number of nodal points on the internal boundary. This is analogous to using spatial Dirac delta weighting functions to enforce compatibility between adjacent distributed substructures. Another possibility exists when certain displacements and rotations at nodal points on the internal boundary make only small contributions to the lower eigenvectors of the assembled structure. In such cases, there is no need to enforce complete compatibility of these displacements and rotations. Of course, exact compatibility between discrete substructures can always be obtained by choosing m_{rs} linearly independent unit weighting vectors.

Numerical Examples

To illustrate the ideas advanced in this paper, we consider two examples, the first representing a distributed system and the second a discrete system.

First, let us consider the thin rectangular nonuniform membrane[13] extending over the closed domain $\bar{D}$ defined by $0\le x\le 9$ and $0\le y\le 5$. The external boundary of the domain consists of the straight lines $x=0,9$ and $y=0,5$ and the membrane is assumed to be clamped along these boundaries. Let us assume that the membrane is divided into three equally-sized substructures with the domains $\bar{D}_1$, $\bar{D}_2$, and $\bar{D}_3$, respectively, where $\bar{D}_1$ is the rectangle $0\le x\le 3$, $0\le y\le 5$, $\bar{D}_2$ is the rectangle $3\le x\le 6$, $0\le y\le 5$, and $\bar{D}_3$ is the rectangle $6\le x\le 9$, $0\le y\le 5$. There exist two internal boundaries S_{12} and S_{23}, where S_{12} is the straight line $x=3$, $0\le y\le 5$ and S_{23} is the straight line $x=6$, $0\le y\le 5$. As substructure admissible functions, we take functions that are separable in the x and y coordinates. To this end, each substructure admissible function $\phi_{sk}(k=1,2,\ldots,N_s;\ s=1,2,3)$ can be written as

$$\phi_{s,N_{sy}(i-1)+j}=\chi_{si}(x)\psi_{sj}(y),$$
$$i=1,2,\ldots,N_{sx};\quad j=1,2,\ldots,N_{sy};\quad s=1,2,3 \tag{18}$$

The functions $\chi_{si}(x)$ and $\psi_{sj}(y)$, given explicitly in Ref. 13, are taken as simple polynomials in x and y and of degrees i and $j+1$, respectively. For convenience, we take $N_{1x}=N_{2x}=N_{3x}=N_x$ and $N_{1y}=N_{2y}=N_{3y}=N_y$, so that

$$N=\sum_{s=1}^{3}N_s=3N_xN_y$$

The disjoint substructures are coupled together via the compatibility conditions (14). Note that in this example $p=1$, so that it is only necessary to consider the displacement compatibility condition (14a). For convenience, the same weighting functions are chosen for each internal boundary, so that $g_{12i}=g_{23i}=g_i(y)(i=1,2,\ldots,M_{12}=M_{23})$. The weighting functions are taken in the form

$$g_i(y)=(y/5)^{i-1} \tag{19}$$

The eigenvalue problem was formulated and solved for various combinations of admissible functions and weighting functions. In the following, we examine the effects on the estimated eigenvalues for the assembled structure of increasing the number of substructure admissible functions and also of increasing the number of substructure weighting functions. To this end, we consider the following cases: 1) $N_x=2$, $N_y=2$, $N=12$, 2) $N_x=3$, $N_y=2$, $N=18$, 3) $N_x=3$, $N_y=3$, $N=27$, 4) $N_x=4$, $N_y=3$, $N=36$ and 5) $N_x=4$, $N_y=4$, $N=48$. The requirement that compatibility conditions at an internal boundary be linearly independent restricts us to the use of at most N_y of the weighting functions (19) for any case. The first three respective eigenvalue estimates for each case are tabulated in Table 1 using 1, 2, and 3 weighting functions at each internal boundary. Note that the estimated eigenvalues in Table 1 decrease going from left to right as more admissible functions are used, and increase going from top to bottom as more weighting functions are used. The first three associated eigenfunctions, for the case when $N_x=N_y=4$ and three weighting functions are used, are displayed in Fig. 1. Note that the geometric compatibility is excellent for these three lower modes.

Next, we consider a discrete system in the form of an unrestrained framework (Fig. 2) consisting of two identical substructures.[15] Each substructure is modeled by the finite-element method, where the substructure finite-element model consists of 34 elements and 29 unrestrained nodal points. For simplicity, we consider only motion of the structure in the z-direction, i.e., motion normal to the plane of the structure.

Table 1 Estimated membrane eigenvalues

$M_{12}=M_{23}$	Estimated eigenvalues	$N_x=N_y=2$	$N_x=3,\ N_y=2$	$N_x=N_y=3$	$N_x=4,\ N_y=3$	$N_x=N_y=4$
1	λ_1	0.461593	0.455267	0.448874	0.448845	0.448817
	λ_2	0.795914	0.788754	0.775652	0.769534	0.769469
	λ_3	1.458914	1.337955	1.265000	1.252535	1.249957
2	λ_1	0.461671	0.455365	0.448980	0.448952	0.448927
	λ_2	0.796409	0.789272	0.776147	0.770019	0.769972
	λ_3	1.579649	1.350837	1.270994	1.258073	1.257883
3	λ_1	–	–	0.450865	0.450855	0.450826
	λ_2	–	–	0.784771	0.778744	0.769972
	λ_3	–	–	1.346336	1.328890	1.328487

Each finite element is capable of bending displacements in the z-direction and of torsion, so that at each nodal point there are three degrees of freedom: the translation in the z-direction, the rotation about the x-axis, and the rotation about the y-axis. Therefore, each substructure finite-element model possesses $n_s = 87$ $(s = 1,2)$ degrees of freedom. The internal boundary S_{12} between the substructures consists of five nodal points equally spaced along the line $x = 0$, $-L_y/2 \leq y \leq L_y/2$, so that the total number of compatibility conditions (9) connecting the two substructures is $m_{12} = 15$. Each substructure is represented by a truncated set of admissible vectors obtained by discretizing admissible functions for a thin rectangular plate free along all of its edges. Considering substructure 1, we take as a set of admissible functions $\phi_i(D)$ for the free plate the functions

$$\begin{aligned}\phi_i(D) = \{ & 1, x/L_x, y/L_y, (y/L_y)^2, (y/L_y)^3, (y/L_y)^4, \\ & (y/L_y)^5, (x/L_x)(y/L_y), (x/L_x)(y/L_y)^2, (x/L_x)^2, \\ & (x/L_x)^3, (x/L_x)^2(y/L_y), (x/L_x)^4, \\ & (x/L_x)^3(y/L_y), (x/L_x)^2(y/L_y)^2, (x/L_x)(y/L_y)^3, \\ & (x/L_x)^5, (x/L_x)^4(y/L_y), (x/L_x)^3(y/L_y)^2, \\ & (x/L_x)^2(y/L_y)^3, (x/L_x)(y/L_y)^4, \ldots \} \end{aligned} \tag{20}$$

which are infinitely-many times differentiable in the domain $0 \leq x \leq L_x$, $-L_y/2 \leq y \leq L_y/2$. The first three of these functions represent rigid-body modes of the free plate and result in rigid-body modes for the discrete substructure. The choice of admissible functions and their order of appearance in the set (20) was determined so that nine linearly independent compatibility conditions (16) can be obtained whenever nine or more admissible vectors, obtained by discretizing these functions, are used to represent the substructure motion. The process of discretizing continuous functions was described earlier and in more detail in Refs. 14 and 15. To represent substructure two, admissible vectors are obtained by discretizing the same functions (20) with x replaced by $(-x)$ and y replaced by $(-y)$. For convenience, we take $N_1 = N_2$, so that $N = N_1 + N_2 = 2N_1 = 2N_2$.

The two disjoint discrete substructures are coupled together via the compatibility conditions (16), where we take the weighting vectors $\boldsymbol{g}_{rs_i}$ in the form of standard unit vectors. Three different intermediate structures defined by the choice of weighting vectors and their number are considered by: 1) using five weighting vectors so as to satisfy compatibility exactly at the nodal point in the center of the internal boundary and to enforce equality of displacements at the two exterior points, 2) using seven weighting vectors so as to enforce equality of rotations about the y-axis at the two exterior points on the internal boundary in addition to the previous five conditions and 3) using nine weighting vectors so as to satisfy compatibility exactly at the center and two exterior nodal points on the internal boundary. The first seven nonzero eigenvalues for each intermediate structure are tabulated in Table 2 using $N_1 = N_2 = 13$, 16, 19, and 21 admissible vectors for each substructure. In addition, three zero eigenvalues associated with the rigid-body modes are obtained in each case. Again, note that the estimated eigenvalues decrease (or at least do not increase) as more substructure admissible vectors are used, and increase (or at least do not decrease) as more weighting vectors are used to enforce geometric compatibility. The first three associated assembled structure eigenvectors, for the case when $N_1 = N_2 = 19$ and nine weighting vectors are used, are displayed in Fig. 3.

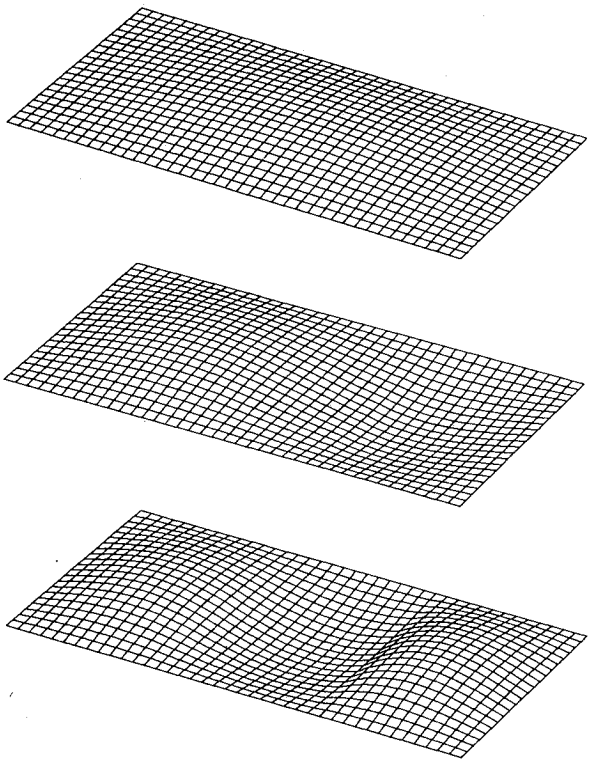

Fig. 1 The first three membrane computed eigenfunctions.

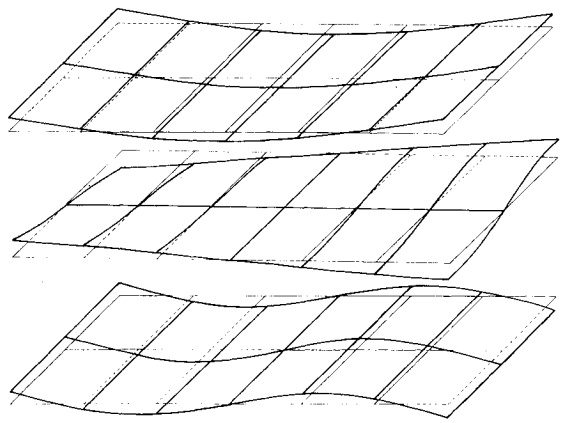

Fig. 3 The first three framework computed eigenvectors.

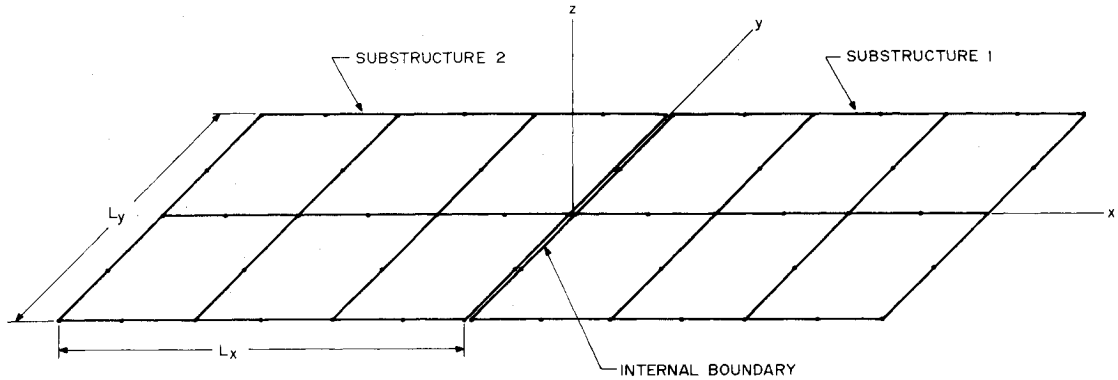

Fig. 2 The discrete model.

Table 2 Estimated discrete structure eigenvalues

No. of weighting vectors	Estimated eigenvalue	$N_1=N_2=13$	$N_1=N_2=16$	$N_1=N_2=19$	$N_1=N_2=21$
5	λ_4	1.390225	1.327956	1.235995	1.234469
	λ_5	5.386810	5.106418	5.020586	5.015776
	λ_6	11.316338	11.292249	11.242735	11.242288
	λ_7	21.057026	18.316971	18.316971	18.316971
	λ_8	39.838631	39.071914	35.809436	35.763506
	λ_9	85.775425	79.283087	63.186588	59.872058
	λ_{10}	143.444421	142.115909	102.642428	95.398833
7	λ_4	1.396658	1.395907	1.393357	1.393262
	λ_5	5.386810	5.106418	5.020586	5.015776
	λ_6	11.316338	11.292249	11.242735	11.242288
	λ_7	30.204161	20.201594	20.144592	20.109250
	λ_8	40.169231	40.144045	39.204337	39.204299
	λ_9	85.775425	79.283087	63.186588	59.872058
	λ_{10}	147.870637	146.553160	118.290916	118.066184
9	λ_4	1.396658	1.395907	1.393357	1.393262
	λ_5	5.386927	5.133477	5.046618	5.046022
	λ_6	11.317336	11.292360	11.242843	11.242373
	λ_7	30.204161	20.201594	20.144592	20.109250
	λ_8	40.169231	40.144045	39.204337	39.204299
	λ_9	85.775567	79.396068	63.276659	59.909863
	λ_{10}	147.870637	146.553160	118.364089	118.084067

Summary and Conclusions

In the general substructure synthesis method presented in this paper, each distributed (discrete) substructure is represented by a given number of substructure admissible functions (vectors). Although substructure eigenfunctions (eigenvectors) can be used as substructure admissible functions (vectors) their use is not really necessary, and quite often not even desirable. This is so because substructure eigenfunctions (eigenvectors) may not be so easy to obtain and, perhaps more importantly, not so easy to work with computationally. To connect together the disjoint substructures, geometric compatibility between substructures is enforced only approximately by using an approach based on weighted residuals in conjunction with weighting functions (vectors). The particular choice of weighting functions (vectors) used and their number defines a fictitious structure, lying between the disjoint structure and the actual structure, called an intermediate structure. Basically all substructure synthesis methods, including those described in Refs. 1-10, replace the actual structure by an intermediate structure (although they do not state so explicitly). The eigenvalues obtained for the intermediate structure approximate the eigenvalues of the original structure but they do not provide upper bounds. For convergence to the eigenvalues of the original structure, it is always necessary to consider two limiting processes, one in which the number of admissible functions (vectors) is increased and the other in which the number of internal boundary weighting functions (vectors) is increased.

Two examples demonstrating the general substructure synthesis are presented: one for a structure composed of distributed substructures and the other for one composed of discrete substructures. Examination of the estimated eigenvalues for both examples, as summarized in Tables 1 and 2, respectively, permits us to conclude that, at least in these examples, simple admissible functions (vectors) are capable of yielding good convergence to a particular intermediate structure. Moreover, a small number of simple weighting functions (vectors) is capable of yielding good convergence to the actual distributed (discrete) structure.

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